Chapter 3
Compact Schemes

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Generalization of Difference Formulas

- In general, a difference approximation to the m^{th} derivative at grid point j can be cast in terms of $q+p+1$ neighboring points as

$$\left(\frac{\partial^m u}{\partial x^m}\right)_j - \sum_{i=-p}^{q} a_i u_{j+i} = er_t \qquad (1)$$

  where the a_i are coefficients to be determined through the use of

- Forward, backward, skewed, or central point operators of any order for any derivative.

Compact Difference Formulas

- A generalization of Eq. 1 can include derivatives at neighboring points, i.e.,

$$\sum_{i=-r}^{s} b_i \left(\frac{\partial^m u}{\partial x^m}\right)_{j+i} - \sum_{i=-p}^{q} a_i u_{j+i} = er_t \qquad (2)$$

- An example of such a formula written on terms of general coefficients a, b, c, d, e is

$$d\left(\frac{\partial u}{\partial x}\right)_{j-1} + \left(\frac{\partial u}{\partial x}\right)_{j} + e\left(\frac{\partial u}{\partial x}\right)_{j+1} - \frac{1}{\Delta x}(au_{j-1} + bu_j + cu_{j+1}) = er_t$$

- Here not only is the derivative at point j represented, but also included are derivatives at points $j-1$ and $j+1$ which also must be expanded using Taylor series about point j.

- Requires generalization of the Taylor series expansion

$$\left(\frac{\partial^m u}{\partial x^m}\right)_{j+k} = \left\{\left[\sum_{n=0}^{\infty} \frac{1}{n!}(k\Delta x)^n \frac{\partial^n}{\partial x^n}\right]\left(\frac{\partial^m u}{\partial x^m}\right)\right\}_j \tag{3}$$

- Derivative terms now have coefficients which must be determined using the Taylor table approach as outlined below.

Taylor Table for Compact Difference Formulas

$$d\left(\frac{\partial u}{\partial x}\right)_{j-1} + \left(\frac{\partial u}{\partial x}\right)_{j} + e\left(\frac{\partial u}{\partial x}\right)_{j+1} - \frac{1}{\Delta x}\left(a u_{j-1} + b u_j + c u_{j+1}\right) = e r_t$$

	u_j	$\Delta x \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \left(\frac{\partial^4 u}{\partial x^4}\right)_j$	$\Delta x^5 \left(\frac{\partial^5 u}{\partial x^5}\right)_j$
$\Delta x \ d\left(\frac{\partial u}{\partial x}\right)_{j-1}$		d	d (-1) $\frac{1}{1!}$	d (-1)^2 $\frac{1}{2!}$	d (-1)^3 $\frac{1}{3!}$	d (-1)^4 $\frac{1}{4!}$
$\Delta x \left(\frac{\partial u}{\partial x}\right)_{j}$		1				
$\Delta x \ e\left(\frac{\partial u}{\partial x}\right)_{j+1}$		e	e (1) $\frac{1}{1!}$	e (1)^2 $\frac{1}{2!}$	e (1)^3 $\frac{1}{3!}$	e (1)^4 $\frac{1}{4!}$
$-a \ u_{j-1}$	$-a$	$-a$ (-1) $\frac{1}{1!}$	$-a$ (-1)^2 $\frac{1}{2!}$	$-a$ (-1)^3 $\frac{1}{3!}$	$-a$ (-1)^4 $\frac{1}{4!}$	$-a$ (-1)^5 $\frac{1}{5!}$
$-b \ u_j$	$-b$					
$-c \ u_{j+1}$	$-c$	$-c$ (1) $\frac{1}{1!}$	$-c$ (1)^2 $\frac{1}{2!}$	$-c$ (1)^3 $\frac{1}{3!}$	$-c$ (1)^4 $\frac{1}{4!}$	$-c$ (1)^5 $\frac{1}{5!}$
$=$ $\Delta x e r_t$	0	0	0	0	0	?

Taylor Table for Compact Difference Formulas

- Maximize the order of accuracy
- Set the the first five columns to zero producing the matrix equation for the coefficients,

$$\begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & -2 & 2 \\ 1 & 0 & -1 & 3 & 3 \\ -1 & 0 & -1 & -4 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Having the solution $[a, b, c, d, e] = \frac{1}{4}[-3, 0, 3, 1, 1]$.

- Under these conditions, the sixth column sums to

$$er_t = \frac{\Delta x^4}{120}\left(\frac{\partial^5 u}{\partial x^5}\right)_j$$

- A 4^{th} order accurate method
- The method can be expressed as

$$\left(\frac{\partial u}{\partial x}\right)_{j-1} + 4\left(\frac{\partial u}{\partial x}\right)_j + \left(\frac{\partial u}{\partial x}\right)_{j+1} - \frac{3}{\Delta x}(-u_{j-1} + u_{j+1}) = O(\Delta x^4)$$

- Obviously, the implementation of such a method requires more explanation.
- Matrix forms of difference schemes will be useful.